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THREE-DIMENSIONAL SUPERSONIC EQUILIBRIUM FLOW OF A GAS AROUND BODIES AT THE ANGLE OF ATTACK */503

O.N.Katskova and P.I.Chushkin

13482

Determination of the purely supersonic zone of a thermodynamically stable gas flow around a three-dimensional body. A method of trigonometric approximation is developed for the case of an arbitrary number of meridional planes. A special variation of the numerical-characteristic method is worked out for the determination of flow parameters. In this technique, the numerical solution is developed by means of bands perpendicular to the axis of the solid. The method is said to ensure second-order accuracy and to make possible the use of a continuous trigonometric approximation, thus minimizing the number of layers and meridional planes to be dealt with.

Author

Section 1. Introduction

The solution of the problem of supersonic flow around bodies of revolution or other three-dimensional bodies moving at an angle of attack through the atmosphere may be most accurately and most effectively obtained by the aid of numerical methods with computers.

This process is usually divided into two stages. The flow around the nose section of the body is first calculated. The flow in this region may either be supersonic (if the body has a pointed conical nose) or transonic (if the body

* Numbers in the margin indicate pagination in the original foreign text.

has a blunt nose).

In the second stage, the pure supersonic region of the flow beyond the nose cone of the body is calculated. The initial data obtained in the first stage of the solution are used in solving the Cauchy problem for the system of hyperbolic-type equations.

Various numerical methods may be used for determining the parameters of a three-dimensional flow in the pure supersonic region. The method of finite differences has been applied to the calculation of three-dimensional supersonic flow around bodies (Bibl.1). A method of solution for this three-dimensional problem by the numerical method of integral relations has also been presented (Bibl.2).

The numerical method of characteristics, which is widely used to solve two-dimensional problems of gas dynamics, is only just beginning to find application in the calculation of three-dimensional supersonic gas flows. A tetrahedral scheme of the three-dimensional method of characteristics has been given (Bibl.3). Other schemes have been discussed (Bibl.4). Although the generalization of the method of characteristics to the three-dimensional case involves no fundamental mathematical problems, the practical execution of the complete three-dimensional method of characteristics on computers does encounter difficulties due to the complex behavior of the characteristic surfaces, the unwieldy computational algorithm and the long programs, which demand a large computer memory.

In this connection, a numerical method that seems promising is one in which, by means of approximations in one of the variables (for instance in the angular variable ψ in the case of cylindrical coordinates), the three-dimensional /504 system of differential equations of the problem can be transformed into a two-dimensional system of differential equations. The latter system will contain a larger number of unknown functions, since it will also contain the values of the approximated functions in all coordinate planes on which they are interpolated (i.e., in the case of cylindrical coordinates, on all the considered meridional planes ψ = const). This system of equations in two variables may be solved by the two-dimensional method of characteristics.

The idea of considering the lines of intersection of the three-dimensional characteristic surfaces with the coordinate planes, instead of these surfaces themselves, has been proposed (Bibl.5). Trigonometric approximations on the variable ψ were first used by Sychev (Bibl.6) for calculating the spatial supersonic flow around bodies of revolution at the angle of attack. In that case, the solution of the problem was reduced to the application of the two-dimensional method of characteristics in two meridional planes, parallel to the oncoming flow.

In the present work, we consider the calculation of the pure supersonic region of flow over bodies with a flow of gas in thermodynamic equilibrium around them. Here, the method of trigonometric approximations with a variable ψ is developed for the case of an arbitrary number of meridional planes. To determine the flow parameters, we developed a special scheme of the method of characteristics in which the numerical solution is constructed by layers perpendicular to the body axes. On each layer the parameters are calculated at certain fixed points, from which two-dimensional characteristics lying in the meridional planes under consideration are laid off in the direction of the preceding layer. This computational scheme yields second-order accuracy and makes it possible to use straight-through trigonometric approximations, which ultimately permits the use of fewer meridional planes and layers.

A brief description of our computational scheme has been given elsewhere (Bibl.2). In a previous report (Bibl.7), we investigated this numerical method of characteristics by layers in the special case of supersonic axisymmetric flow around a body of revolution in a flow of a perfect gas at zero angle of attack.

Two papers have recently been published (Bibl.8, 9) applying a combination of the two-dimensional method of characteristics and the method of finite differences to the problem of three-dimensional supersonic flow around a body. A number of examples of calculated flows have been briefly mentioned elsewhere (Bibl.8). In the solution developed by the author (Bibl.8), the third variable is already eliminated (as it is in our own method) in the initial three-dimensional system of equations by the aid of an approximate representation of the corresponding partial derivatives. However, the mentioned computational method (Bibl.8) has certain shortcomings. Its accuracy is of the first order; in calculation by layers, the two-dimensional characteristics are here produced from the preceding layer in the direction of the new layer, and no conversions are made to allow for it. The partial derivatives to be eliminated are determined from the difference of the corresponding functions on two adjacent interpolation planes, and a large number of such planes must therefore be introduced. Owing to the construction of the solution in ordinary Cartesian or cylindrical coordinates (without "straightening" the surfaces of the body and the shock wave), the calculation of the approximate partial derivatives close to these surfaces becomes extremely complicated. /505

Section 2. The Equations of the Problem

Let a supersonic uniform flow of a nonviscous thermally nonconducting gas in a state of thermodynamic equilibrium flow around a certain body at a velocity $\mathbf{V}_\infty$ and an angle of attack α. Assume, for simplicity, that the body has a plane of symmetry parallel to the velocity vector of the relative flow; it may, in particular, be a solid of revolution. Consider the calculation of the pure supersonic three-dimensional flow region between the surface of the body and the shock wave. We will assume that the body is rather smooth, and confine the calculation to cases in which no secondary shock waves appear within the flow region.

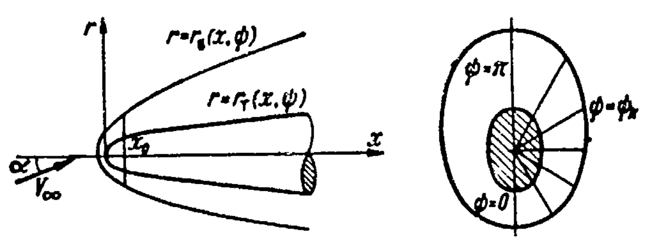

Fig.1

Let us use the cylindrical coordinate system x, r, ψ, fixed in the body (Fig.1). Let the origin of coordinates be located at the leading edge of the body, and let the x axis be directed along the body axis. Let the value $\psi = 0$ correspond to the windward side, and $\psi = \pi$ to the leeward side. By virtue of the symmetry of flow, it will be sufficient to consider the region $0 \leq \psi \leq \pi$.

In calculating the purely supersonic flow region, it is necessary to solve the Cauchy problem for a system of differential equations of hyperbolic type, depending on the three variables x, r, ψ. Let us assume the initial Cauchy data (i.e., the fields of all the required gas-dynamic functions) to be known in some plane $x = x_0$, passing through the supersonic region.

Let us take the system of equations of gas dynamics describing the steady-state equilibrium three-dimensional flow of gas in the following form:

$$\rho a^2 \nabla \mathbf{V} + \mathbf{V}\nabla p = 0, \quad \rho(\mathbf{V}\nabla)\mathbf{V} + \nabla p = 0, \quad \mathbf{V}\nabla s = 0, \tag{2.1}$$

where **V**, p, ρ, s and a denote respectively the velocity vector, pressure, density, entropy, and velocity of sound.

To determine the temperature T, we use the thermodynamic equality

$$dT = qds + gdp, \tag{2.2}$$

where

$$q = \frac{T}{h_T'}, \qquad g = \frac{1}{h_T'}\left(\frac{1}{\rho} - h_p'\right),$$

Here, h denotes the enthalpy, while the subscripts and primes denote the corresponding partial derivatives. The density ρ and the enthalpy h in a gas flow in thermodynamic equilibrium are connected with the temperature T and the pressure p by the well-known relations: /506

$$\rho = \rho(T, p), \qquad h = h(T, p), \tag{2.3}$$

which, for the case of air, are represented in analytic form by the aid of approximations [cf. (Bibl.10)]. To calculate the velocity of sound a, we use the relation

$$a = \left[\rho_p' + \frac{\rho_T'}{h_T'}\left(\frac{1}{\rho} - h_p'\right)\right]^{-1/2}. \tag{2.4}$$

Equations (2.1) - (2.4) thus form a complete system describing the equilibrium flows of a real gas. In this system, we will consider all quantities to be dimensionless and take as the characteristic quantities some linear dimension (for instance the oblateness or nose radius, r_*), the velocity and density of the relative flow V_∞ and ρ_∞, and the gas constant R. Then the dimensionless pressure will be given by the quantity $\rho_\infty V_\infty^2$, the temperature by the quantity V_∞^2/R, the enthalpy by the quantity V_∞^2, and the entropy by the quantity R.

For an ideal gas with a constant adiabatic index $\varkappa$, eqs.(2.1) - (2.2) keep their form, while eqs.(2.3) - (2.4) in these dimensionless variables reduce to the following:

$$\rho = p^{1/\varkappa} \exp\left(\frac{1-\varkappa}{\varkappa} s\right), \qquad T = \frac{p}{\rho}, \qquad h = \frac{\varkappa}{\varkappa - 1} T, \qquad a^2 = \varkappa \frac{p}{\rho}. \tag{2.5}$$

Let us now write eqs.(2.1) in cylindrical coordinates:

$$\begin{aligned}
&\rho a^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \psi} + \frac{v}{r}\right) + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial r} + \frac{w}{r}\frac{\partial p}{\partial \psi} = 0,\\
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial \psi} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,\\
&u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} + \frac{w}{r}\frac{\partial v}{\partial \psi} - \frac{w^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,\\
&u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial r} + \frac{w}{r}\frac{\partial w}{\partial \psi} + \frac{vw}{r} + \frac{1}{r\rho}\frac{\partial p}{\partial \psi} = 0,\\
&u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial r} + \frac{w}{r}\frac{\partial s}{\partial \psi} = 0,
\end{aligned} \tag{2.6}$$

where u, v, w are, respectively, the values of the velocity components $\mathbf{V}$ in the directions of the x, r, ψ coordinates.

Thus, the system (2.6) defines five fundamental functions: the velocity components u, v, w, the pressure p, and the entropy s. The density ρ, the temperature T, the enthalpy h, and the velocity of sound a which also enter that system, in the general case of equilibrium flow of the gas, are expressed in terms of these fundamental functions by the aid of the equalities (2.2) - (2.4).

The boundary conditions for the system (2.6), besides the Cauchy conditions for $x = x_0$, are conditions for the required shock wave $r = r_w(x, \psi)$, which are the well-known relations for a strong gas-dynamic separation of flow, and the no-flow conditions on the given surface of the body $r = r_b(x, \psi)$.

Section 3. Method of Solving the Problem /507

For the numerical solution of this problem of three-dimensional supersonic flow around a body, let us use a method in which the initial system of three-dimensional differential equations will be reduced to a system of two-dimensional differential equations for a larger number of functions.

First, let us substitute the variable r by a new independent variable ξ:

$$\xi = \frac{1}{\varepsilon}[r - r_b(x, \psi)],$$

where $\varepsilon = r_w(x,\psi) - r_b(x,\psi)$. Obviously, $\xi = 0$ corresponds to the body and $\xi = 1$ to the shock wave, and

$$r = \varepsilon\xi + r_b. \tag{3.1}$$

In this case, the transformation formulas are of the form

$$\begin{aligned}
\frac{\partial F(x, r, \psi)}{\partial x} &= \frac{\lambda}{\varepsilon}\frac{\partial F(x, \xi, \psi)}{\partial \xi} + \frac{\partial F(x, \xi, \psi)}{\partial x}, \\
\frac{\partial F(x, r, \psi)}{\partial r} &= \frac{1}{\varepsilon}\frac{\partial F(x, \xi, \psi)}{\partial \xi}, \\
\frac{\partial F(x, r, \psi)}{\partial \psi} &= \frac{r\mu}{\varepsilon}\frac{\partial F(x, \xi, \psi)}{\partial \xi} + \frac{\partial F(x, \xi, \psi)}{\partial \psi},
\end{aligned}$$

where

$$\lambda = -\xi(r_{wx}' - r_{bx}') - r_{bx}', \quad \mu = -\frac{1}{r}[\xi(r_{w\psi}' - r_{b\psi}') + r_{b\psi}'],$$

and, for a solid of revolution $r = r_b(x)$, and $r'_{b\,\psi} = 0$.

After passing to the variables x, ξ, ψ, the system (2.[illegible]) can be written in the following form:

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{1}{\varepsilon}\left(\lambda\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \xi} + \mu\frac{\partial w}{\partial \xi}\right) + \frac{1}{\rho a^2}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial \xi}\right) + \Phi_1 = 0, \\
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial \xi} + \frac{1}{\rho}\left(\frac{\partial p}{\partial x} + \frac{\lambda}{\varepsilon}\frac{\partial p}{\partial \xi}\right) + \Phi_2 = 0, \\
&u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial \xi} + \frac{1}{\rho\varepsilon}\frac{\partial p}{\partial \xi} + \Phi_3 = 0, \\
&u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial \xi} + \frac{\mu}{\rho\varepsilon}\frac{\partial p}{\partial \xi} + \Phi_4 = 0, \\
&u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial \xi} + \Phi_5 = 0,
\end{aligned} \tag{3.2}$$

where

$$\nu = \frac{1}{\varepsilon}(\lambda u + v + \mu w), \qquad \Phi_1 = \frac{1}{r}\left(\frac{\partial w}{\partial\psi} + \frac{w}{\rho a^2}\frac{\partial p}{\partial\psi} + v\right),$$
$$\Phi_2 = \frac{w}{r}\frac{\partial u}{\partial\psi}, \qquad \Phi_3 = \frac{w}{r}\left(\frac{\partial v}{\partial\psi} - w\right) \tag{3.3}$$
$$\Phi_4 = \frac{1}{r}\left(w\frac{\partial w}{\partial\psi} + \frac{1}{\rho}\frac{\partial p}{\partial\psi} + vw\right), \qquad \Phi_5 = \frac{w}{r}\frac{\partial s}{\partial\psi}.$$

The system (3.2) contains derivative functions in all three variables /508 x, ξ, ψ. To eliminate the derivatives with respect to ψ from this system, let us approximate the corresponding gas-dynamic functions with respect to this variable. For this purpose, in the region $0 \le \psi \le \pi$, let us consider $\ell + 1$ meridional planes equidistant in ψ, $\psi = \psi_k = k\pi/\ell$ $(k = 0, 1, \ldots, \ell)$. Let us further represent the odd F (i.e., w) and the even $\tilde{F}$ (i.e., r, u, v, p, s) functions in ψ by the interpolational trigonometric polynomials

$$F(x, \xi, \psi) = \sum_{k=1}^{l-1} a_k(x, \xi)\sin k\psi, \qquad \tilde{F}(x, \xi, \psi) = \sum_{k=0}^{l} b_k(x, \xi)\cos k\psi. \tag{3.4}$$

The coefficients $a_k(x,\xi)$ and $b_k(x,\xi)$ are expressed in terms of the values of the functions being interpolated on the meridional planes considered, and are of the form

$$a_k = \sum_{j=1}^{l-1} c_{kj}F_j, \qquad b_k = \sum_{j=0}^{l} d_{kj}\tilde{F}_j,$$

in which the subscripts j denote the values of F and $\tilde{F}$ at $\psi = \psi_j = j\pi/\ell$. The numerical coefficients c_{kj}, d_{kj} for $\ell = 2, 3, 4$ are given in the Table.

For the approximations (3.4) adopted, the values of the derivatives with respect to ψ in the planes $\psi = \psi_k$ are given by the expressions

$$\left(\frac{\partial F}{\partial\psi}\right)_k = \sum_{j=1}^{l-1} e_{kj}\tilde{F}_j, \qquad \left(\frac{\partial \tilde{F}}{\partial\psi}\right)_k = \sum_{j=0}^{l} f_{kj}F_j, \tag{3.5}$$

where, obviously, $(\partial\tilde{F}/\partial\psi)_0 = (\partial\tilde{F}/\partial\psi)_\ell = 0$. The numerical coefficients e_{kj}, f_{kj}

for $i = 2, 3, 4$ are given in the same Table.

Thus, after application of the approximations (3.4), the three-dimensional system (3.2) becomes a system of differential equations in the two independent variables x and ξ. However, in this case the values of the required functions on the meridional planes $\psi = \psi_k$ will enter into the system (through the functions Φ_i containing derivatives with respect to ψ). Consequently this two-dimensional system must be considered on each of these meridional planes.

We find that, under the condition

$$u^2 + \frac{(v+\mu w)^2}{1+\mu^2} > a^2$$

the two-dimensional system (3.2) will be of the hyperbolic type and have two families of real characteristics. The differential equations of the characteristics of the first and second families may be represented in the following form:

$$\frac{d\xi}{dx} = \frac{1}{\varepsilon}\left(\lambda + \frac{u^2\zeta + a^2\beta}{u^2 - a^2}\right) \equiv A_1, \tag{3.6}$$

$$\frac{d\xi}{dx} = \frac{1}{\varepsilon}\left(\lambda + \frac{u^2\zeta - a^2\beta}{u^2 - a^2}\right) \equiv A_2, \tag{3.7}$$

where /509

$$\zeta = \frac{1}{u}(v + \mu w), \qquad \beta = \sqrt{\frac{u^2\zeta^2}{a^2} + (1+\mu^2)\left(\frac{u^2}{a^2} - 1\right)}. \tag{3.8}$$

The differential compatibility equations for the characteristics of the first and second families may be written, respectively, in the following forms:

$$d\zeta + Kdp + Ldx = 0, \tag{3.9}$$

$$d\zeta - Jdp + Ndx = 0, \tag{3.10}$$

where

$$J = K = \frac{\beta}{\rho u^2},$$

$$L = \frac{1}{u^2}\left[\Phi_1 \frac{u a^2(\zeta+\beta)}{u^2-a^2} - \Phi_2 \frac{u^2\zeta + a^2\beta}{u^2-a^2} + \Phi_3 + \mu\Phi_4 - uw\frac{d\mu}{dx}\right],$$

$$N = \frac{1}{u^2}\left[\Phi_1 \frac{u a^2(\zeta-\beta)}{u^2-a^2} - \Phi_2 \frac{u^2\zeta - a^2\beta}{u^2-a^2} + \Phi_3 + \mu\Phi_4 - uw\frac{d\mu}{dx}\right].$$

Table

$l=2$

$c_{11}=1, \quad d_{00}=d_{02}=d_{20}=d_{22}=\frac{1}{4}, \quad d_{01}=d_{10}=-d_{12}=-d_{21}=\frac{1}{2}, \quad d_{11}=0,$

$e_{01}=-e_{21}=1, \quad e_{11}=0, \quad f_{10}=-f_{12}=-\frac{1}{2}, \quad f_{11}=0.$

$l=3$

j	1	2
k	$\sqrt{3}\,c_{kj}$	
1	1	1
2	1	−1
k	$2\sqrt{3}\,e_{kj}$	
0	6	−2
1	−1	3
2	−3	1
3	2	−6

j	0	1	2	3
k	$6d_{kj}$			
0	1	2	2	1
1	2	2	−2	−2
2	2	−2	−2	2
3	1	−2	2	−1
k	$2\sqrt{3}\,f_{kj}$			
1	−3	1	3	−1
2	1	−3	−1	3

$l=4$

j	1	2	3
k	$4c_{kj}$		
1	$\sqrt{2}$	2	$\sqrt{2}$
2	2	0	−2
3	$\sqrt{2}$	−2	$\sqrt{2}$
k	$2e_{kj}$		
0	$2+2\sqrt{2}$	−2	$-2+2\sqrt{2}$
1	−1	$2\sqrt{2}$	−1
2	−2	0	2
3	1	$-2\sqrt{2}$	1
4	$2-2\sqrt{2}$	2	$-2-2\sqrt{2}$

j	0	1	2	3	4
k	$8d_{kj}$				
0	1	2	2	2	1
1	2	$2\sqrt{2}$	0	$-2\sqrt{2}$	−2
2	2	0	−4	0	2
3	2	$-2\sqrt{2}$	0	$2\sqrt{2}$	−2
4	1	−2	2	−2	1
k	$2f_{kj}$				
1	$-1-\sqrt{2}$	1	2	−1	$-1+\sqrt{2}$
2	1	$-2\sqrt{2}$	0	$2\sqrt{2}$	−1
3	$1-\sqrt{2}$	1	−2	−1	$1+\sqrt{2}$

In addition, characteristic properties are also exhibited by another /510 family of lines, defined by the equations

$$\frac{d\xi}{dx} = \frac{1}{\varepsilon}(\lambda + \zeta) = A_4. \tag{3.11}$$

By analogy to the axisymmetric flow, these lines will be arbitrarily termed "streamlines". Along such "streamlines" the following relations will hold:

$$Bdv - dw + Cdx = 0, \tag{3.12}$$

$$du + Gdv + Pdp + Qdx = 0, \tag{3.13}$$

$$ds + Edx = 0, \tag{3.14}$$

where

$$B = \mu, \qquad C = \frac{1}{u}(\mu\Phi_3 - \Phi_4), \qquad G = \zeta,$$

$$P = \frac{1}{\rho u}, \qquad Q = \frac{1}{u}(\Phi_2 + \zeta\Phi_3), \qquad E = \frac{\Phi_5}{u}.$$

We note that by combining eqs.(3.12) and (3.13) we can obtain still another relation which will be valid along the "streamlines":

$$\frac{1}{2}d(u^2 + v^2 + w^2) + \frac{1}{\rho}dp + \frac{1}{u}(u\Phi_2 + v\Phi_3 + w\Phi_4)dx = 0.$$

The differential equations of the characteristics and of the "streamlines" and the resultant relations along these lines, for the system (3.2), are of a form closely resembling that of the corresponding equations and relations for the system of equations of gas dynamics in the axisymmetric case. If, in eqs.(3.6) - (3.14), we set $w = 0$ and $\mu = 0$, it is easy to derive the equations for axisymmetric flow in the variables x, ξ in the form given elsewhere (Bibl.7) for an ideal gas.

Section 4. Numerical Scheme of the Method of Characteristics

We shall now integrate the system of hyperbolic-type differential equations (3.2) by the numerical method of characteristics. In connection with the use of an approximation of the type (3.4) in solving the problem, it will be expedient to use a special scheme of the method of numerical characteristics instead of the standard scheme. In this special scheme, the calculation is performed by layers bounded by planes x = const, and the characteristics on each new layer are produced in the direction toward the preceding layer, through /511
selected points with values of ξ = const.

To solve the problem, the differential equations of the characteristics and of the "streamlines" and the relations along these lines [eqs.(3.6) - (3.14)] are considered on each of the $\ell + 1$ meridional planes $\psi = \psi_k$ $(k = 0, 1, \ldots, i)$ introduced. These equations and relations are then represented in the form of finite differences according to a scheme of second-order accuracy. The resultant system of finite equations, which contains the values of the gas-dynamic functions on all meridional planes, is solved by iteration.

We will discuss in some detail the computational scheme of the method of characteristics. As already assumed above, on some layer $x = x_0$, located between the body and the shock wave in the hyperbolic region, let there be known the fields of the velocity components u, v, w, of the pressure p, of the entropy s, and of the temperature T, as well as the radius of the shock wave r_w and the derivative r_{wx}'. Of course, let the geometry of the body, i.e., the functions $r_b(x,\psi)$, $r_{b\psi}'(x,\psi)$, $r_{bx}'(x,\psi)$, also be prescribed. Let us select $\ell + 1$, the number of the meridional planes $\psi = \psi_k (k = 0, 1, \ldots, \ell)$, and in each plane let us consider $n + 1$ points with values $\xi = \xi_m = m/n$ $(m = 0, 1, \ldots, n)$; let us take these values $\xi = \xi_m$ as the same for all $\psi = \psi_k$. From the known fields of the fundamental functions, we then obtain the initial Cauchy data at all points with the coordinates x_0, ξ_s, ψ_k. Let us now determine the values of the fundamental functions on the new layer $x = x_0 + \Delta x$ at points with the same ξ_s, ψ_k.

The flow parameters on the new layer are calculated in the following sequence: First we calculate the points on the shock wave ($\xi = 1$) at all values of $\psi = \psi_k$. Then we determine the gas-dynamic parameters at the interior points of the region, using the iteration method to solve the system of corresponding finite-difference equations for the individual value of $\xi = \xi_s$, and simultaneously for all $\psi = \psi_k$. Moving from the shock wave toward the body, we calculate the flow parameters at all considered points along the interior lines corresponding to successively decreasing values of $\xi = \xi_s$. Finally, for all $\psi = \psi_k$, we calculate the points along the contour of the cross section of the body ($\xi - 0$).

Let us now consider in detail the solution scheme of the individual elementary problems of the method of characteristics: determination of the flow parameters at the points on the shock wave, at interior points, and at points on the body surface.

1. <u>Point on the Shock Wave</u>

Here the discontinuity relations are used to find the gas dynamic functions immediately beyond the shock wave. In the case of equilibrium flow of the gas, let us take these relations in the form

$$p - p_\infty = \left(1 - \frac{1}{\rho}\right)\cos^2(\mathbf{n}, \mathbf{V}_\infty), \tag{4.1}$$

$$h - h_\infty = \frac{\Delta p}{2}\left(1 + \frac{1}{}\right), \tag{4.2}$$

/512

$$u - u_\infty = -\Delta p \frac{\cos(\mathbf{n}, \mathbf{x})}{\cos(\mathbf{n}, \mathbf{V}_\infty)}, \qquad v - v_\infty = -\Delta p \frac{\cos(\mathbf{n}, \mathbf{r})}{\cos(\mathbf{n}, \mathbf{V}_\infty)},$$
$$w - w_\infty = -\Delta p \frac{\cos(\mathbf{n}, \psi)}{\cos(\mathbf{n}, \mathbf{V}_\infty)}$$

where the subscript ∞ denotes the parameters of the relative flow $\Delta p = p - p_\infty$ while, for the angle between the normal $\mathbf{n}$ to the surface of the shock wave and the velocity vector of the relative flow $\mathbf{V}_\infty$, we have

$$\cos(\mathbf{n}, \mathbf{V}_\infty) = \frac{\delta}{\sqrt{1 + r_{wx}'^2 + (r_{w\psi}'/r)^2}}, \tag{4.3}$$

where

$$\delta = r_{wx}' \cos\alpha + \sin\alpha\cos\psi + \frac{r_{w\psi}'}{r_w}\sin\alpha\sin\psi.$$

Obviously, in our dimensionless variables, the velocity component of the relative flow normal to the shock wave, will be

$$V_{n\infty} = \cos(\mathbf{n}, \mathbf{V}_\infty). \tag{4.4}$$

These relations on the shock wave must be associated with the well-known relations (2.3) between ρ, h, p, and T in equilibrium flow as well as with eq.(2.2) which defines the entropy s in terms of these four functions. Let us rewrite the expressions for the velocity components beyond the shock wave as follows:

$$\begin{aligned} u &= \cos\alpha - \Delta p \frac{r_{xw}'}{\delta}, \\ v &= -\sin\alpha\cos\psi + \Delta p \frac{1}{\delta}, \\ w &= \sin\alpha\sin\psi - \Delta p \frac{r_{w\psi}'}{r_w \delta}. \end{aligned} \tag{4.5}$$

For the case of an ideal gas, when neither the temperature T nor the enthalpy h need be considered, the relations on the shock wave are simplified.

In that case, the pressure, density, and entropy beyond the shock wave are calculated by the explicit formulas

$$\Delta p = \frac{2}{\varkappa + 1}\left(V_{n\infty}^2 - \frac{1}{M_\infty^2}\right),$$

$$\rho = \frac{\Lambda_\infty^2 V_{n\infty}^2}{1 - [(\varkappa - 1)/(\varkappa + 1)]\Lambda_\infty^2(1 - V_{n\infty}^2)}, \tag{4.6}$$

$$s = \frac{1}{\varkappa - 1}\ln\frac{p}{\rho^\varkappa},$$

while eqs.(4.5) for the velocity components retain their form. In eqs.(4.6) the velocity $\Lambda_\infty = V_\infty/a_{cr}$, referred to the critical velocity of sound a_{cr}, is expressed in terms of the Mach number M_∞:

$$\Lambda_\infty^2 = \frac{(\varkappa + 1)M_\infty^2}{2 + (\varkappa - 1)M_\infty^2}.$$

The points located along the shock wave on the new layer $x = x_0 + \Delta x$ are calculated by successive approximations. Consider first any of these points /513 3 (Fig.2a) in one of the meridional planes $\psi = \psi_k$, for instance in the plane $\psi = 0$. Here we assign the value of the derivative r_{wx}' close to the value of the derivative at the preceding point 5 on the shock wave. The radius of the shock wave at the point 3 is then calculated by the formula

$$r_{w3} = r_{w5} + {}^1/_2[(r_{wx}')_3 + (r_{wx}')_5]\Delta x$$

and the value of $\cos(\mathbf{n}, \mathbf{V}_\infty)$ by eq.(4.3).

In the case of equilibrium flow, at the point 3 the values of pressure p_3 and temperature T_3 are prescribed, and ρ_3 and h_3 are found from eq.(2.3). The value of T is varied with the selected p_3 in such manner that eq.(4.1) is satisfied with the required accuracy. In turn, the value of p_3 is varied such that [at the value of T_3 already selected by the condition (4.1)] eq.(4.2) is satisfied. In this process of selecting the values of p_3 and T_3, any desired interpolation or iteration schemes can be used. After completing the selection of p_3 and T_3, the velocity components are determined by eqs.(4.5), and the value of the entropy s_3 by eq.(2.2) written in finite-difference form along the shock wave:

$$s_3 = s_5 + \frac{1}{g}[T_3 - T_5 - g(p_3 - p_5)]. \tag{4.7}$$

In the case of an ideal gas, the process of selecting p_3 and T_3 is no longer necessary, since eqs.(4.6) permit a direct calculation of p_3, ρ_3, and s_3 from the prescribed value of $r_{w\xi}$ at the point 3.

Further, starting from the point 3 we produce the characteristics of the first family I in the direction of the preceding layer $x = x_0$, and represent eqs.(3.6) and (3.9) in finite-difference form:

$$\xi_1 = 1 - A_1 \Delta x, \tag{4.8}$$

$$\zeta_3 - \zeta_1 + K(p_3 - p_1) + L\Delta x = 0. \tag{4.9}$$

From eq.(4.8) we find the coordinate ξ_1 of the point 1 on the layer $x = x_0$.

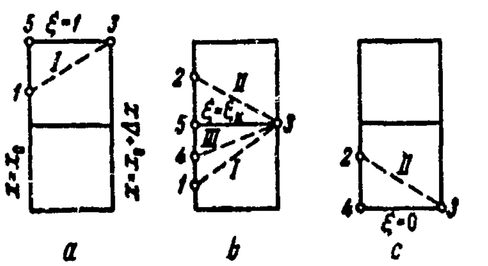

Fig.2

Using quadratic interpolation at three adjacent points on this known layer, we then determine the values of all fundamental functions at the point 1, calculating the radius r_1 by eq.(3.1). We then check whether the equality (4.9) is satisfied. In general, it will not be satisfied for arbitrarily chosen values of $r_{w\xi}$ at the point 3, meaning that such a value of $r_{w\xi}$ must be changed for the $\psi = \psi_k$ considered. This completes the calculation of one approximation for the point 3 located on the shock wave in one meridional plane $\psi = \psi_k$. /514

This computational process is conducted successively to compute, in first approximation, other points 3 on the shock wave for all remaining $\psi = \psi_k$, for which the corresponding values of $r_{w\xi}$ are prescribed for each $\psi = \psi_k$.

After completing the cycle of computations in first approximation, we proceed with an analogous computational process in the second and succeeding approximations for all $\psi = \psi_k (k = 0, 1, \ldots, \ell)$ until values of $r_{w\xi}$ have been selected such that all the equalities of the type of eq.(4.9) are simultaneously satisfied, with the required degree of accuracy, for each $\psi = \psi_k$.

We note that the coefficients g and q in eq.(4.7) are the averages for the points 3 and 5, while the coefficients A_1, K, L entering into eqs.(4.8) and (4.9) are averaged for the points 1 and 3 (only in the first approximation is the coefficient A_1 used for the point 3). The derivatives of r_w, u, v, w, and p with respect to ψ, which enter into λ, μ, $\cos(\mathbf{n}, \mathbf{V}_\infty)$, and L, are taken in the first approximation at the points 3, the same as at the corresponding points 5 on the layer $x = x_0$. In the following approximations, these derivatives are determined by the aid of the interpolation formulas (3.5) from the values of the functions calculated in the preceding approximation.

2. Interior Point of the Region

We pass now to a description of the iteration process of calculating the interior points of the flow region which are located on the layer $x = x_0 + \Delta x$, at one value of $\xi = \xi_*$ and a series of values of $\psi = \psi_k$ $(k = 0, 1, \ldots, \ell)$.

Considering at first any one meridional plane $\psi = \psi_k$, let us produce, from the computational point 3, the characteristics of the first family I and of the second family II and the "streamline" III, which intersect the known plane x = = x_0 at the points 1, 2, and 4, in sequence (Fig.2b). The coordinates of these points are determined from the finite-difference equations that follow from eqs.(3.6), (3.7), and (3.11):

$$\xi_i = \xi_3 - A_i \Delta x \qquad (i = 1, 2, 4)$$

By quadratic interpolation on the layer x = x_0, we find at these three points $\xi = \xi_i$ the quantities u, v, w, p, T and at the point 4 also the entropy s.

Further, on the basis of the compatibility relations (3.9) and (3.10) written in finite-difference form, we express p_3 and ζ_3:

$$\begin{gathered} p_3 = p_1 + \frac{1}{K+J}[\zeta_1 - \zeta_2 - J(p_1 - p_2) + (N - L)\Delta x], \\ \zeta_3 = \zeta_1 - K(p_3 - p_1) - L\Delta x, \end{gathered} \tag{4.10}$$

while from eqs.(3.14) and (2.2), used along the "streamline", we obtain the following formulas for the entropy s_3 and the temperature T_3:

$$s_3 = s_4 - E\Delta x, \tag{4.11}$$
$$T_3 = T_4 + q(s_3 - s_4) + g(p_3 - p_4). \tag{4.12}$$

Let us now take the finite-difference analogs of the two remaining relations (3.12) and (3.13), valid along the "streamlines", and eq.(3.8) for the quantity ζ. These three equalities permit determining the three velocity components at the point 3: /515

$$\begin{gathered} w_3 = \frac{B\{\zeta_3[u_4 - P(p_3 - p_4) - Q\Delta x] - v_4\} + (1 + G\zeta_3)(w_4 + C\Delta x)}{1 + G\zeta_3 + B\mu_3}, \\ u_3 = \frac{1}{1 + G\zeta_3}[u_4 - P(p_3 - p_4) - Q\Delta x + G(v_4 + \mu_3 w_3)], \\ v_3 = u_3\zeta_3 - \mu_3 w_3. \end{gathered} \tag{4.13}$$

Equations (4.10) - (4.13) make it possible to calculate successively, at the point 3, the quantities p, s, T, u, v, w. The density ρ, the enthalpy h, and the velocity of sound a, entering into these expressions, are found by the aid of eqs.(2.3) and (2.4), and the radius r by eq.(3.1).

In the case of an ideal gas, when the temperature T and the enthalpy h are excluded from the fundamental functions, eq.(4.12) is omitted, while the density ρ and the velocity of sound a are found in terms of p and s from eqs.(2.5).

The above-described computational procedure is performed successively for all points corresponding to the prescribed $\xi = \xi_m$ and to all considered $\psi = \psi_k$. In this way we calculate the first iteration for solving the system of difference equations. An analogous computational cycle is repeated in the following iterations, permitting the determination of all required functions with the required accuracy. The true signs are usually found after three iterations.

In the first iteration, the coefficients A_i and the derivatives of u, v, w, p, s with respect to ψ that enter the quantities Φ_i of eq.(3.3) are used for calculating the points 3, according to their values at the corresponding points 5 on the preceding layer. In the system of difference equations (4.10) - (4.13) the coefficients K, L, J, N, E, g, q, B, C, G, P, Q (as well as the coefficients A_i in the succeeding iterations) are determined, respectively, from points 3 and 1, 3 and 2, 3 and 4. The above-mentioned derivatives with respect to ψ in the second and following iterations are determined by the interpolation formulas (3.5), using the data from the preceding iteration.

3. Point on the Body Surface

The calculation of the points on the body surface ($\xi = 0$) in many respects is similar to the calculation of the interior points of the flow region, and is performed by successive iterations for all values of $\psi = \psi_k$. We will describe this computational procedure for one plane $\psi = \psi_k$ and for the case of equilibrium flow.

Let us construct the region of characteristics of the second family II, passing through the considered points 3 on the surface of the body (Fig.2c). The coordinate ξ_2 of the point 2 of the intersection of this characteristic with the layer $x = x_0$ will be $\xi_2 = -A_2\Delta x$. At this point, the values of all fundamental functions are determined, as is conventional, by quadratic interpolation. From eq.(3.10), taken in finite-difference form, we obtain

$$p_3 = p_2 + \frac{1}{j}(\zeta_3 - \zeta_2 + N\Delta x). \tag{4.14}$$

The no-flow condition on the body surface, i.e., the condition that the normal velocity component is to vanish, leads at the point 3 to the equality

$$\lambda_3 u_3 + v_3 + \mu_3 w_3 = 0. \tag{4.15}$$

Hence, bearing in mind the definition (3.8) for the quantity ζ, we find /516

$$\zeta_3 = -\lambda_3. \tag{4.16}$$

It is likewise obvious that the line 3-4 on the surface of the body is a "streamline"; consequently the relations (3.12) - (3.14) are valid along it.

Writing these latter equalities in finite-difference form, as well as eq.(2.2), will yield, in addition to eqs.(4.14) - (4.16), a system of equations for calculating all fundamental quantities at the point 3 on the surface of the body. The formulas for determining the functions s_3, T_3, u_3, v_3, and w_3 here coincide with the corresponding eqs.(4.11) - (4.13) for an interior point of the flow region, while the pressure p_3 and the quantity ζ_3 are calculated from eqs.(4.14) and (4.16). The density ρ_3, the enthalpy h_3, and the velocity of sound a_3 are found from eqs.(2.3) - (2.4).

The coefficients entering the difference equations are averaged here in the same way as in calculating an interior point. The changes in the computational scheme for the case of an ideal gas will be analogous.

After calculating the flow parameters for the entire layer $x = x_0 + \Delta x$, the accuracy of the solution can be checked. For this purpose, using the approximations adopted for the fundamental functions, the discharge and energy of the gas are found by integration over this layer, after which we determine whether or not the corresponding conditions of conservation are satisfied.

The computational grid, i.e. the number $\ell + 1$ meridional planes $\psi = \psi_k$ ($k = 0, 1, \ldots, \ell$) and the number $n + 1$ points $\xi = \xi_m$ ($m = 0, 1, \ldots, n$) in each plane, is selected according to the required accuracy. The use of trigonometric approximations and relatively smooth variation of the functions with ψ in the case of smooth bodies at moderate angles of attack make it possible, for practical purposes, to confine the calculation to the range $\ell = 4 - 6$, while n is taken in the range of 25 - 50. The choice of the spacing Δx is governed by stability considerations.

We note one more consequence of eq.(3.14). If, in the initial cross section, the entropy on the body surface was everywhere the same (i.e., $\partial s/\partial\psi = 0$), it will also remain constant along the entire body (naturally, on condition that no new shock waves are formed on the surface of the body). This property also follows from a consideration of the flow around the surface of the body within the framework of rigorous theory.

Section 5. Calculation Examples

This method has been used in calculating, for illustrative purposes, various

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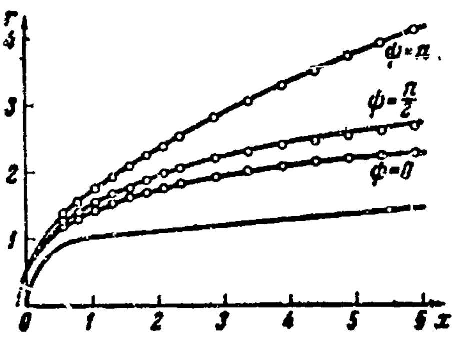

Fig.3

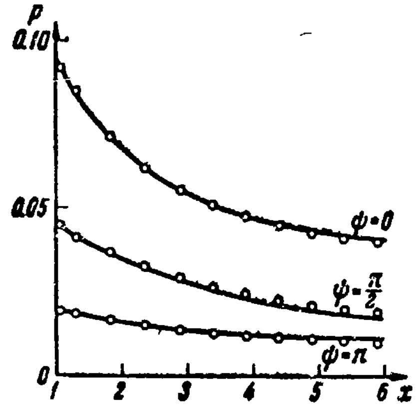

Fig.4

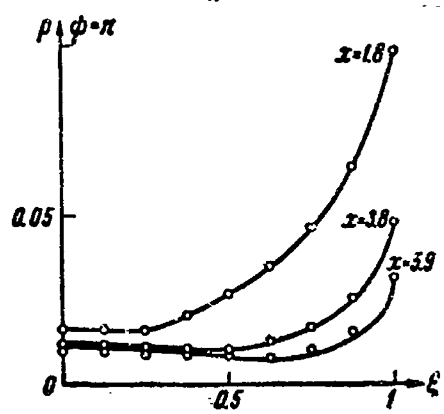

Fig.5

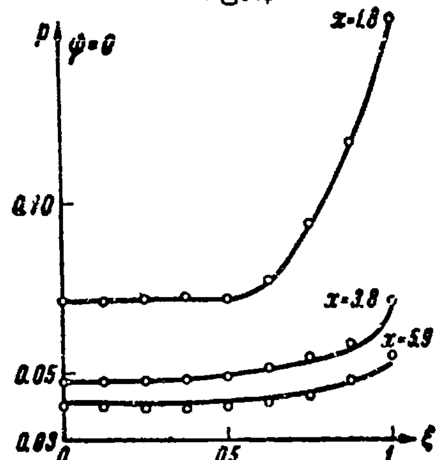

Fig.6

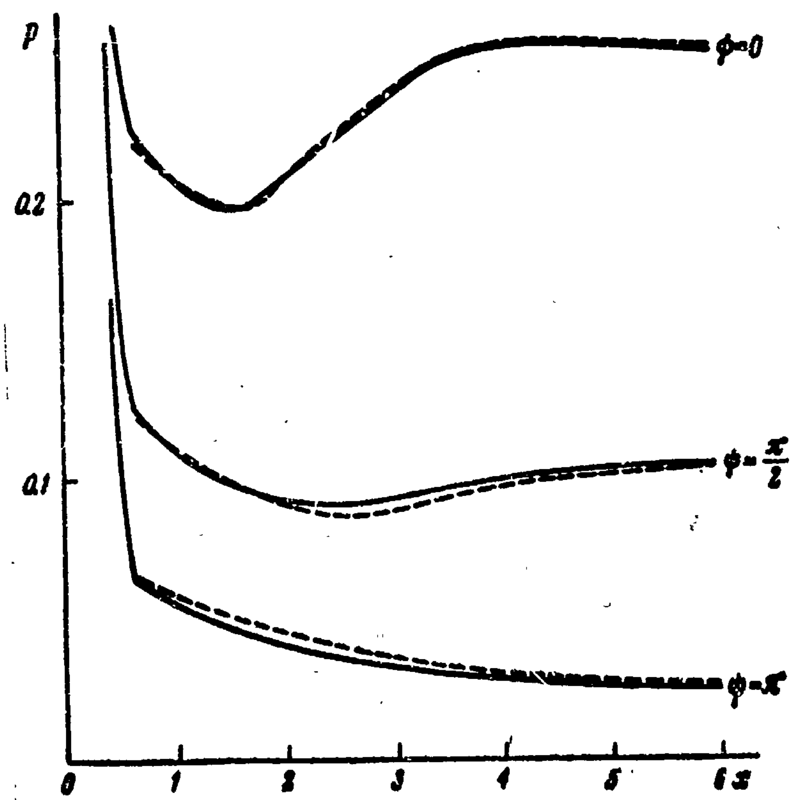

Fig.7

cases of supersonic flow around blunted cones at an angle of attack in a stream of ideal air ($\varkappa = 1.4$). The cones were of circular cross section, with a spherically oblate nose. The cases differed in the vertex half-angle ω, the Mach number M_∞ of the relative flow, and the angle of attack α. The initial Cauchy data in the hyperbolic region were taken from available calculations of supersonic flow around a sphere (Bibl.11).

A few results of the calculations of three-dimensional supersonic flow around a truncated cone $\omega = 5^0$ are given by Figs.3 - 6, referring to the case of $M_\infty = \infty$ and $\alpha = 10^0$. In all these graphs, the linear dimensions are given /518 in multiples of the nose radius, and the origin of the cylindrical system of coordinates is located at the leading edge of the body.

The individual variants were calculated with different computational grids. The number n, determining the number of points in each meridional plane, was taken as n = 25. A comparison with the corresponding calculations at n = 50 showed that this value of n yielded satisfactory accuracy. The number ℓ of meridional planes was also varied in the calculations. All cases were calculated for five meridional planes, $\ell = 4$ (solid lines in the graphs), and some cases for three meridional planes, $\ell = 2$ (circles in the graphs).

In Fig.3 the form of the shock wave is plotted in sections ψ = const, and Fig.4 gives the pressure distribution along the surface of the body (on the linear segment of the generatrix).

Figs.5 - 6 give the variation in pressure between the shock wave and the surface of the body along the layers x = const for $\psi = 0$ and $\psi = \pi$.

Finally, in Fig.7 the pressure distributions calculated for $\ell = 4$ on a cone $\omega = 20^0$ for $M_\infty = \infty$ and $\alpha = 10^0$ are compared with the corresponding results (but at $M_\infty = 20$) obtained in (Bibl.12) by the finite-difference method (Bibl.1) and shown by the dashed line in the diagram.

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